Expansion of a Bose-Einstein Condensate in the Presence of Disorder

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Abstract

Expansion of a Bose-Einstein condensate (BEC) is studied, in the presence of a random potential. The expansion is controlled by a single parameter, $(\mu\tau_{eff}/\hbar)$, where μ is the chemical potential, prior to the release of the BEC from the trap, and τ_{eff} is a transport relaxation time which characterizes the strength of the disorder. Repulsive interactions (nonlinearity) facilitate transport and can lead to diffusive spreading of the condensate which, in the absence of interactions, would have remained localized in the vicinity of its initial location.

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There is a number of recent experiments on the interplay between nonlinearity and disorder [1, 2, 3, 4, 5, 6]. In [1, 2, 3, 4] the expansion of a BEC, in the presence of a random potential, has been studied. In [6] the authors study propagation of a light beam through a dielectric sample, with a random refraction index, under linear and nonlinear conditions. Under certain conditions both types of experiments can be described with the Gross-Pitaevskii equation. For the BEC problem the necessary conditions are specified, e.g., in [7]. For optical waves the Gross-Pitaevskii equation emerges within the paraxial approximation (see [6] and references therein). The equation is:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\triangle\Psi + V(\mathbf{r})\Psi + g|\Psi|^2\Psi, \tag{1}$$

where m is the atom mass, $g = 4\pi a\hbar^2/m$ is the interaction parameter related to the scattering length a, $V(\mathbf{r})$ is the random potential and the wave function is normalized to the total number of atoms,

$$\int d\mathbf{r}|\Psi|^2 = N. \tag{2}$$

We have written the equation in the BEC context and will use the appropriate language throughout the paper (a brief translation into the language of optics will be made at the end). The random potential can be generated optically (a laser speckle pattern, [8]) or it can be due to some foreign (impurity) atoms, with which the BEC interacts [9]. Eq. (1) is applicable in any spatial dimension $d = 1, 2, 3$. In the $2d$ case, to which most of this paper is devoted, the condensate can move only in the (x, y)-plane, whereas the motion in the z-direction is frozen. For this geometry $\mathbf{r}$ represents the in-plane coordinate and $\triangle$ is the corresponding $2d$ Laplacian. The interaction parameter for the $2d$ problem is $g = (8\pi)^{1/2}(\hbar^2 a/ma_z)$, where a_z is the oscillator length in the z-direction [7]. We assume positive g, i.e., repulsive interactions. Eq.(1) has to be solved with the initial condition

$$\Psi(\mathbf{r}, t = 0) = F(\mathbf{r}), \tag{3}$$

where F($\mathbf{r}$) represents the initial shape of the condensate, i.e., the equilibrium shape in the trap prior to its release. It is assumed that at $t = 0$ the trap is suddenly switched off and the expansion, described by Eq. (1), starts.

We will be interested in strong nonlinearity in the sense that, initially, almost the entire energy of the condensate is due to interactions. In this case, and in the absence of the random potential ($V(\mathbf{r}) = 0$), the expansion is known to occur in two stages [10, 11, 7]: a rapid "explosion", followed by expansion linear in time. The explosive stage takes time of order $t_0 = 1/\omega$, where ω is the frequency of the parabolic trap, from which the BEC is released. We assume that the trap is isotropic (for the $2d$ case the isotropy pertains, of course, only to the (x, y)- plane but not to the z-direction). During the time t_0 the condensate, roughly, doubles in size and most of its interaction energy gets converted into the flow energy. The second stage corresponds to times larger than t_0, when the non-linearity becomes weak and the expansion is accurately described by the linear Schrödinger equation. This observation suggests a simple, approximate approach to the problem of expansion in the presence of disorder: If the disorder is sufficiently weak (see the criterion below), then the first stage of the expansion is dominated by strong nonlinearity, while the disorder can be neglected. In the second stage the non-linearity can be neglected, so that one is left with the linear Schrödinger equation, in the presence of disorder. This idea has been proposed in [12], which appeared when the present work was in progress (in [12] the 1d case was considered, whereas we are interested mostly in the 2d and 3d case). Thus, the evolution of the BEC at the second stage of the expansion amounts to a diffusion process in the weakly disordered potential.

At the end of the first stage of the expansion, i.e., at time of order t_0, the condensate

can be roughly described by a wave packet [10, 11, 7]

$$\Psi(r, t_0) \simeq F(r) exp(ir^2/a_0^2), \tag{4}$$

where $a_0 = (\hbar/m\omega)^{1/2}$ is the oscillator size of the (in-plane) trap. The function $F(r)$, on a qualitative level, can be taken the same as the initial wave function in the trap: indeed, during the time t_0 the condensate has only expanded by a factor of 2 or so, preserving its shape of the inverted parabola. The essential thing that happens during the time t_0 is that the wave function acquires a phase, which accounts for the flow energy. Since the velocity of the condensate is proportional to the gradient of the phase in Eq.(4), it follows that the velocity is proportional to r, which is the main feature of the ballistically expanding condensate. The wave function in Eq.(4) serves as the initial condition for the second stage of expansion, controlled by the linear Schrödinger equation, in the random potential $V(\mathbf{r})$. To facilitate the calculations below, we shall approximate the initial function by a Gaussian, $F(r) = Aexp(-r^2/2R_0^2)$, where $A = (N/\pi R_0^2)^{1/2}$ is a normalization factor and R_0 characterizes the initial size of the condensate. This replacement of the inverted parabola by a Gaussian does not affect the evolution of the wave packet (4) in any significant way.

A clear separation into the two stages can be made only if the disorder is sufficiently weak [12]. This means that during the initial, "explosive" stage there is no scattering of the condensate on the random potential, i.e., this initial stage is ballistic. This requirement is satisfied if the transport mean free path l is larger than $2\pi/k$, where k is the wave number of the wave (particle) in question. The wave packet in Eq.(4) contains a broad range of k's, with a Gaussian tail for $k > k_0$, where $k_0 = R_0/a_0^2$. The value of l separates the spatial Fourier components of the packet into two groups. The components with $kl >> 1$ will evolve by a diffusion process, whereas those with $kl < 1$ will be localized by disorder and, thus, will stay in the vicinity of the initial location of the packet. (In $2d$ also the large-k components will eventually get localized, but for those the localization length is exponentially large so that there is a lot of room for the diffusive propagation). In order to clearly observe the diffusive behavior one must take $l \gg k_0^{-1}$. This is a rather weak condition, because k_0^{-1} is of the order of the healing length ξ which is the smallest macroscopic length in the problem. (Note that we assume that the bottom of the trap, from which the condensate was released, coincides with the average value of the random potential; if the bottom were raised sufficiently with respect to that value, then even the small-k components would have enough energy to diffuse away.) To ensure that the great majority of the Fourier components would evolve by diffusion we choose, somewhat arbitrarily, $l \approx R_0$, so that the hierarchy of length scales is

$$\xi \ll a_0 \ll R_0 < l. \tag{5}$$

Note that, since $R_0^2 \simeq \mu/m\omega^2$ and $\xi^2 \simeq \hbar^2/m\mu$, the ratios ξ/a_0 and a_0/R_0 are of the same order, $(\hbar\omega/\mu)^{1/2}$, where μ is the chemical potential of the BEC before it was released from the trap.

Let us now present the calculation of the time evolution of a packet, whose initial shape is defined in (4), with $F(r)$ being a Gaussian, i.e.,

$$\Psi(r, t = 0) = (N/\pi R_0^2)^{1/2} exp(-r^2/2R_0^2) exp(ir^2/2a_0^2), \tag{6}$$

where t_0 in the argument of Ψ has been replaced by 0. This is because we are studying now the time evolution starting from t_0 (and on time scale much larger than t_0). Decomposing $\Psi(r, t = 0)$ into Fourier components, $\tilde{\Psi}(k)$, we obtain

$$\tilde{\Psi}(k) = (N/\pi R_0^2)^{1/2} b^2 exp(-b^2 k^2/2), \tag{7}$$

where the complex length b is defined as

$$\frac{1}{b^2} = \frac{1}{R_0^2} - \frac{i}{a_0^2}. \tag{8}$$

The quantity $\mid \tilde{\Psi}(k) \mid^2$ describes the density of the particles in $\mathbf{k}$-space. More precisely, $dn(k) = \mid \tilde{\Psi}(k) \mid^2 d^2k$ is the number of particles in the interval d^2k, or in the interval d^2v_k in the velocity space ($v_k = \hbar k/m$). The distribution dn_k, emerging from the "explosion", will evolve by diffusion, due to scattering from the random potential $V(\mathbf{r})$. One must keep in mind that the diffusion coefficient, $D(k) = \frac{1}{2} v(k) l(k)$ (as well as the mean free path l), depend on the velocity, so that the group $dn(k)$ of particles, after time t, will spread into a cloud

$$dn_{\mathbf{k}}(\mathbf{r}, t) = \mid \tilde{\Psi}(k) \mid^2 d^2k \frac{1}{4\pi D(k) t} exp(-\frac{r^2}{4D(k)t}). \tag{9}$$

This expression has to be integrated over $\mathbf{k}$, to obtain the BEC shape, $\mid \Psi(r, t) \mid^2$, at time t. To perform this integration we have to specify the dependence of l, or of the mean transport time τ, on k. For a Gaussian white noise potential, τ is known to be inverse proportional to the density of states $\nu(k)$ [13]. Since, in $2d$, $\nu = const$, we have $D(k) = \frac{1}{2} v^2(k) \tau = \frac{1}{2} \frac{\hbar^2 k^2}{m^2} \tau$, with τ being a constant specifying the strength of the random potential. Integration over k of the expression (9) yields

$$\mid \Psi(r, t) \mid^2 = \frac{N}{2\pi D_0 t} K_0(\frac{r}{\sqrt{D_0 t}}), \tag{10}$$

where K_0 is the zeroth order modified Hankel function and D_0 is the diffusion coefficient corresponding to $k = k_0$, i.e., $D_0 = \frac{1}{2} v_0^2 \tau = \frac{1}{2} \frac{\hbar^2 k_0^2}{m^2} \tau$. The mean square size of the condensate grows as $2D_0 t$.

Eq.(10) describes the shape of the condensate in the diffusion regime. It is quite remarkable that this shape depends on the single quantity D_0. It is instructive to rewrite D_0 in terms of the chemical potential μ of the BEC in the trap, prior to its expansion. Using $k_0^2 = R_0^2/a_0^4 = 2m\mu/\hbar^2$, we obtain $D_0 = \mu\tau/m$. Thus, the shape (for a given m) is determined by a single dimensionless parameter $\mu\tau/\hbar$. The situation turns out to be

analogous to a disordered Fermi system, where $(E_F\tau/\hbar) \simeq k_F l$ is the only parameter which determines the transport properties of a disordered metal, with Fermi energy E_F [13]. The necessary condition for the diffusive regime described above is that the parameter $(\mu\tau/\hbar) \gg 1$. When the disorder increases and $\mu\tau/\hbar$ approaches unity the diffusion is inhibited and the BEC gets localized. (In $2d$, even for $(\mu\tau/\hbar) \gg 1$ spreading by diffusion will eventually stop and localization is expected to set in. The localization length, however, is exponentially large, $L_{loc} \sim exp(\mu\tau/\hbar)$.) Instead of changing the disorder, one can tune the interaction strength, i.e., the nonlinearity (for instance, using a Feshbach resonance), and observe a crossover from localization (weak interactions) to diffusion (strong interaction).

In the derivation of Eq.(10) we have ignored the fact that the small-k components of the initial wave packet (6) cannot propagate by diffusion, as was already discussed above. To account for this fact one should integrate the expression (9) with a lower cutoff, k_{min}. The cutoff is determined by the condition $kl(k) \simeq 1$, i.e., $k_{min} \simeq (m/\hbar\tau)^{1/2}$. Although it is not possible then to derive a close analytic expression for $\mid \Psi(r,t) \mid^2$, the qualitative picture is clear: most of the wave packet will evolve according to Eq.(10), but a small portion will remain localized near the origin. Furthermore, we have neglected the weak localization effects, which influence the large-k components and renormalize (by a small amount) the corresponding diffusion coefficient $D(k)$.

Our calculation can be extended to three dimensions. Eq.(9) remains the same, with d^3k instead of d^2k and with

$$\mid \tilde{\Psi}(k) \mid^2 = \frac{N}{(k_0\sqrt{\pi})^3} e^{-k^2/k_0^2}. \tag{11}$$

The main difference with the two-dimensional case is that in $3d$ the density of states, $\nu(k)$, is proportional to k. This leads to a linear dependence of the diffusion coefficient on k (instead of the square dependence, in $2d$). The linear dependence complicates integration over k, and we were not able to obtain a close expression for $\mid \Psi(r,t) \mid^2$. The essential thing, however, is that the spread of $\mid \Psi(r,t) \mid^2$ is still determined by the single quantity, $D_0 = \frac{1}{3} v_0 l = \frac{1}{3} v_0^2 \tau_0$, where $\tau_0 \equiv \tau(k_0)$ (note that, unlike in $2d$, τ now depends on k). Thus, again, the behavior of the condensate is controlled by the single parameter $(\mu\tau_0/\hbar)$, and diffusion is possible only if this parameter is large. We will not dwell here on the diffusion regime but rather mention the interesting possibility of an interaction-induced Anderson transition. The point is that, in $3d$, there is a critical value, c, such that for $\mu\tau_0/\hbar$ smaller than c the BEC remains localized. Therefore, by increasing the interaction strength, one can transfer the condensate from the localized regime into the extended one.

Let us explain this crossover in more detail. For conceptual simplicity it is useful not to create disorder in the trap itself but only somewhat away. Then, in the absence of interactions μ would be small, of order $\hbar\omega$, and the condensate released from the trap would not be able to penetrate the disordered region (we assume that $\hbar\omega$ is smaller than the mobility edge E_c). With the increase of interactions μ, as well as the size of

the condensate R_0, increase and the wave function, after the first (ballistic) stage of expansion, develops oscillations (see Eq.(4)). The number of oscillations grows with μ, so that for larger μ the wave packet in Eq.(4) will contain more high energy components. The components with energy larger than E_c will be able to penetrate the disordered region. Thus, under increase of interactions, larger and larger portions of the condensate will be transferred "over the mobility edge" and will diffuse away.

The specific dependence of l (or τ) on k is determined by the properties of the random potential. In our calculation a Gaussian, white noise potential has been taken. For a correlated potential the dependence of l on k will change (see, for instance, [14, 15]). Let us return to the $2d$ case and assume a Gaussian random potential, with zero mean, and with a correlation function $< V(\mathbf{r})V(\mathbf{r}') >= V_0^2 exp(- \mid \mathbf{r} - \mathbf{r}' \mid^2 / R_c^2)$, where V_0 and R_c characterize, respectively, strength and range of the random potential. For this case a close expression for the transport mean free path l can be obtained [14]:

$$\frac{1}{l} = \frac{\pi}{k}(\frac{mV_0R_c}{\hbar^2})^2 G(k^2R_c^2/2), \tag{12}$$

where the function $G(x)$ is given in terms of the modified Bessel functions as $G(x) = exp(-x)[I_0(x) - I_1(x)]$. For $kR_c \ll 1$ (the white noise limit) l is proportional to k. In the opposite limit of a smooth random potential ($kR_c \gg 1$) l is proportional to k^4. $D(k)$ in (9) becomes a complicated function of k and no closed expression for $\mid \Psi(r,t) \mid^2$ can be obtained. The detailed shape of the expanding cloud is changed, as compared with the white noise case, Eq.(10). The overall features, however, are preserved. For instance, the mean square size of the cloud is still proportional to $D_{eff}t$, where the effective diffusion coefficient depends now on R_c (in addition to V_0 and k_0). Thus, the rough picture of the expansion (but not the fine details) is universal and depends on the single quantity, D_{eff}. This statement pertains to short-range correlations, rapidly decaying beyond the distance R_c (the correlation radius). It has been emphasized already in the original work of Anderson [16] that long range correlations in the randomness change the transport picture completely. The "one-dimensional" speckles, designed in [1, 2, 3, 4], do exhibit long range correlations, which has a significant effect on spreading of the $1d$ condensate [12]. It is not clear how long range correlations will affect the diffusive spreading in 2 or 3 dimensions.

Let us briefly comment on the $1d$ case. There is some numerical work on the subject [17, 18] claiming that even weak non-linearity (although still above a certain threshold) can lead to complete delocalization. We should emphasize that weak nonlinearity effects are not addressed in the present work. We assumed strong nonlinearity, initially, and no nonlinearity after the first, explosive stage of the expansion, when almost all interaction energy had already been converted into flow. More recent numerics [19, 20] concentrated on strong nonlinearity, as appropriate for the experimental conditions [1, 2, 3, 4], and identified various regimes of $1d$ localization, depending on the degree of disorder. In $1d$ the localization length, L_{loc}, is of the same order as the mean free path l, so that there is no

room for diffusion and our results do not apply. However, the classification of the disorder as "weak" or "strong", depending on the value of the parameter $k_0 l \simeq (\mu\tau_0/\hbar)$, remains meaningful. If this parameter is large, then the details of the randomness, such as the correlation radius of the random potential, are not important (single parameter scaling [21]). When this parameter approaches unity, one enters the regime of strong localization where details become important and single parameter scaling is violated [22]. For a correlated Gaussian potential, $< V(x)V(x') >= V_0^2 f(\mid x - x' \mid /R_c)$, the inverse localization length, for a given wave number k, is of the order of [23] $L_{loc}^{-1} \simeq (mV_0/\hbar^2 k)^2 \tilde{f}(kR_c)$, where $\tilde{f}(kR_c)$ is the Fourier transform of some (short range) correlation function $f(\mid x - x' \mid /R_c)$. Note that $\tilde{f}(kR_c)$ has units of length, i.e., it can be written as $R_c g(kR_c)$. For $kR_c \ll 1$, $g(kR_c) \approx 1$ (the white noise limit). In this limit L_{loc} is proportional to $k^2/V_0^2 R_c$. For $kR_c \gg 1$, L_{loc} rapidly increases, due to the decrease of the function $\tilde{f}(kR_c)$, so that the "weak disorder condition", $kL_{loc} \gg 1$, becomes less restrictive.

As has been mentioned at the beginning, Eq.(1) describes also propagation in nonlinear optics, in the paraxial (scalar wave) approximation. A monochromatic beam propagates in the z-direction and spreads in the transverse, (x, y)-plane. To translate Eq.(1) into the language of optics, one should make the following replacements: $t \Longrightarrow z, \hbar \Longrightarrow (c/\omega), m \Longrightarrow 1, V(\mathbf{r}) \Longrightarrow (-\delta n(x,y)/n_0)$, where ω is the frequency of the wave, c is its speed (in the medium), n_0 is the average refraction index, and δn is its fluctuating part (it is crucial that δn does not depend on z). Fluctuations of the refraction index in the transverse plane can lead to "transverse localization" [24] of the wave. The authors of [6] have recently observed this phenomenon, as well as the interplay between nonlinearity and disorder. The self-focusing (self-defocusing) in optics corresponds to attractive (repulsive) interactions in the BEC. Thus, the results presented in this paper applies to optics as well, for a self-defocusing nonlinearity.

In conclusion, the diffusive spreading of a BEC, in the presence of a random potential, has been studied. The spreading is controlled by a single parameter, which can be conveniently written as $(\mu\tau_{eff}/\hbar)$. Here τ_{eff} is some effective relaxation time which depends on the properties of the random potential, as well as on μ (i.e., on $k_0 = \sqrt{2m\mu}/\hbar$). Diffusive spreading is possible only if this parameter is large. There is a clear parallel with the single parameter scaling in disordered electronic systems. There are also essential differences between the two problems. In disordered conductors (at low temperatures) only the electrons in the vicinity of the Fermi energy participate in transport, whereas spreading of the BEC involves a broad range of wave numbers k (we considered only the case when the center of mass of the condensate is fixed, i.e., the condensate spreads but does not move as a whole). Strong repulsive non-linearity, considered in this paper, has a large effect on the BEC spreading and serves as a strong "delocalizing factor".

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